

Applying Hodge theory to detect Hamiltonian flows

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Abstract

We prove that when Hodge theory survives on non-compact symplectic manifolds, a compact symplectic Lie group action having fixed points is necessarily Hamiltonian, provided the associated almost complex structure preserves the space of harmonic one-forms. For example, this is the case for complete Kähler manifolds for which the symplectic form has an appropriate decay at infinity. This extends a classical theorem of Frankel for compact Kähler manifolds to complete non-compact Kähler manifolds.

1 Introduction

The study of fixed points of dynamical systems and group actions is a classical topic studied in geometry. A seminal result of T. Frankel [6] states that if a symplectic circle action on a compact connected Kähler manifold has fixed points, then it must be Hamiltonian. The present paper builds on Frankel's ideas to study whether this striking result persists under some reasonable conditions for possibly non-Kähler, non-compact symplectic manifolds. The non-compact case is of special interest in dynamical systems.

Loosely speaking, the goal of this paper is to prove that *when Hodge theory survives on non-compact symplectic manifolds and the space of harmonic one-forms is preserved by an associated almost complex structure*, the existence of a fixed point for a symplectic action of a compact Lie group forces the action to be Hamiltonian. This has particularly strong implications for complete Kähler manifolds.

All manifolds in this note are paracompact and boundaryless.

1.1 Main Theorem

Let (M, ω) be a symplectic manifold. The triple (ω, g, \mathbf{J}) is a *compatible triple* on (M, ω) if g is a Riemannian metric and \mathbf{J} is an almost complex structure (i.e., a vector bundle automorphism $\mathbf{J}: TM \rightarrow TM$ satisfying $\mathbf{J}^2 = -\text{Identity}$) such that $g(\cdot, \cdot) = \omega(\cdot, \mathbf{J}\cdot)$. For the following theorem recall that the standard construction of a compatible triple from a symplectic form immediately extends to the G -invariant case.

Let G be a Lie group with Lie algebra \mathfrak{g} . Suppose that G acts on M by symplectomorphisms (i.e., diffeomorphisms which preserve the symplectic form). Any element $\xi \in \mathfrak{g}$ generates a vector field ξ_M on M , called the *infinitesimal generator*, given by $\xi_M(x) := \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi) \cdot x$, where $\exp: \mathfrak{g} \rightarrow G$ is the exponential map and $x \in M$. The G -action on (M, ω) is said to be *Hamiltonian* if there exists a smooth equivariant map $\mu: M \rightarrow \mathfrak{g}^*$, called the *momentum map*, such that for all $\xi \in \mathfrak{g}$ we have $\mathbf{i}_{\xi_M} \omega := \omega(\xi_M, \cdot) = \mathbf{d}\langle \mu, \xi \rangle$, where $\langle \cdot, \cdot \rangle: \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ is the duality pairing. For example, if $G \simeq (S^1)^k$, $k \in \mathbb{N}$, is a torus, the existence of such a map μ is equivalent to the exactness of the one-forms $\mathbf{i}_{\xi_M} \omega$ for all $\xi \in \mathfrak{g}$. In this case the obstruction of the action to being Hamiltonian lies in the first de Rham cohomology group of M . The simplest example of a S^1 -Hamiltonian action is rotation of the sphere S^2 about the polar axis; see Figure 1.1. The flow lines of the infinitesimal generator defining this action are the latitude circles.

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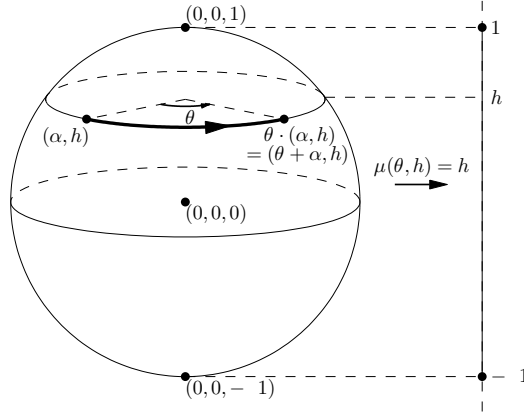


Figure 1.1: Momentum map for the S^1 -action on S^2

We denote by L^2_ρ the L^2 -Hilbert space of square integrable functions relative to the measure ρ .

Theorem 1. *Let G be a compact Lie group acting on a symplectic manifold (M, ω) by means of symplectomorphisms. Let (ω, g, \mathbf{J}) be a G -invariant compatible triple. Let λ be a measure on M such that the Radon-Nikodym derivative of λ relative to the Riemannian measure is a bounded function on M and denote by δ_λ the formal adjoint of \mathbf{d} relative to the L^2_λ inner product.*

Assume that each L^2_λ closed one-form decomposes L^2_λ -orthogonally as a sum of the differential of a L^2_λ smooth function and a harmonic L^2_λ one-form (i.e., in the joint kernel of \mathbf{d} and δ_λ) and that each cohomology class of a closed one-form in L^2_λ has a unique harmonic representative. If \mathbf{J} preserves harmonic one-forms and the G -action has fixed points on every connected component, then the action is Hamiltonian.

As far as the authors know this is the first instance in which the relation between the existence of fixed points and the Hamiltonian character of the G -action has been studied for non-compact manifolds. All assumptions of the theorem, with possibly the exception of the existence of fixed points, hold for compact Kähler manifolds.

Remark 2 Note that the theorem is implied by the case $G = S^1$ using a standard argument based on the fact that every point of a compact Lie group lies on a maximal torus. We recall the proof. First, note that if the theorem holds for S^1 , then it also holds for any torus $\mathbb{T}^k := (S^1)^k$, $k \in \mathbb{N}$, since the momentum map of the product of two Hamiltonian actions is the sum of the two momentum maps. Now let G be an arbitrary compact Lie group whose symplectic action on M has at least a fixed point. If $\xi \in \mathfrak{g}$, then $\exp \xi$ necessarily lies in a maximal torus and the restriction of the action to the torus has fixed points. Since the conclusion of the theorem holds for symplectic torus actions, it follows that this restricted action has an invariant momentum map. In particular, $\mathbf{d}i_{\xi_M} \omega = \mathbf{d}f^\xi$ for some $f^\xi \in C^\infty(M)$, a relation valid for every $\xi \in \mathfrak{g}$. Using a basis $\{e_1, \dots, e_r\}$ of \mathfrak{g} , we define a new map $\mu : M \rightarrow \mathfrak{g}^*$ by $\mu^\xi := \xi^1 f^{e_1} + \dots + \xi^r f^{e_r}$, where $\xi = \xi^1 e_1 + \dots + \xi^r e_r$. We clearly have $\mathbf{d}i_{\xi_M} \omega = \mathbf{d}\mu^\xi$ which proves that $\mu : M \rightarrow \mathfrak{g}^*$, defined by the requirement that its ξ -component is μ^ξ for each $\xi \in \mathfrak{g}$, is a momentum map of the G -action. Since G is compact, one can construct out of μ an *equivariant* momentum map (see, e.g., [17, Theorem 11.5.2]), which proves that the action is Hamiltonian. \circ

Example 3 The assumption that the action has fixed points in Theorem 1 is essential. For example, the S^1 -action on \mathbb{T}^2 given by $e^{2i\varphi} \cdot (e^{2i\theta_1}, e^{2i\theta_2}) := (e^{2i\theta_1}, e^{2i(\theta_2 + \varphi)})$ is a symplectic action on a Kähler manifold which is free and hence has no fixed points. Recall that the range of the derivative of the momentum map

at a given point equals the annihilator of the symmetry algebra of that point (the Reduction Lemma). If $G = S^1$, since the manifold is compact, the momentum map must have critical points which shows that the action in this example does not admit a momentum map. \oslash

1.2 Consequences in the Kähler case

If the manifold is Kähler, the associated complex structure automatically preserves the space of harmonic one-forms. Since for compact manifolds one always has the Hodge decomposition for the measure ω^n , $2n = \dim M$, we immediately conclude the following statement.

Corollary 4. *Let (M, ω) be a compact symplectic G -space, where G is a compact Lie group. Assume that the space of harmonic one-forms is invariant under the complex structure (which always holds if M is a Kähler manifold). If the G -action has fixed points on every connected component of M then it is Hamiltonian.*

A consequence of the proof of Theorem 1 is the following result whose proof is given at the end of Section 2.

Corollary 5. *Let (M, ω) be a $2n$ -dimensional complete connected Kähler G -space, where G is a compact Lie group. If the contraction of ω with all infinitesimal generators of the action is in $L^2_{\omega^n}$ and the G -action has fixed points then it is Hamiltonian.*

1.3 Frankel's Theorem and further results

The first result concerning the relationship between the existence of fixed points and the Hamiltonian character of the action is Frankel's celebrated theorem [6] stating that if the manifold is compact, connected, and Kähler, $G = S^1$, and the symplectic action has fixed points, then it must be Hamiltonian (note that $J\mathcal{H} \subset \mathcal{H}$ holds, see [23, Cor 4.11, Ch. 5]). Frankel's influential work has inspired subsequent research. McDuff [18, Proposition 2] has shown that any symplectic circle action on a compact connected symplectic 4-manifold having fixed points is Hamiltonian. However, the result is false in higher dimensions since she gave an example ([18, Proposition 1]) of a compact connected symplectic 6-manifold with a symplectic circle action which has fixed points (formed by tori), but is not Hamiltonian. If the S^1 -action is semifree (that is, it is free off the fixed point set), then Tolman and Weitsman [22, Theorem 1] have shown that any symplectic S^1 -action on a compact connected symplectic manifold having fixed points is Hamiltonian. Feldman [5, Theorem 1] characterized the obstruction for a symplectic circle action on a compact manifold to be Hamiltonian and deduced the McDuff and Tolman-Weitsman theorems by applying his criterion. He showed that the Todd genus of a manifold admitting a symplectic circle action with isolated fixed points is equal either to 0, in which case the action is non-Hamiltonian, or to 1, in which case the action is Hamiltonian. In addition, any symplectic circle action on a manifold with positive Todd genus is Hamiltonian. For additional results regarding aspherical symplectic manifolds (i.e. $\int_{S^2} f^*\omega = 0$ for any smooth map $f : S^2 \rightarrow M$) see [11, Section 8] and [16]. As of today, there are no known examples of symplectic S^1 -actions on compact connected symplectic manifolds that are not Hamiltonian but have fixed points.

For higher dimensional Lie groups, less is known. Giacobbe [7, Theorem 3.13] proved that a symplectic action of a n -torus on a $2n$ -dimensional compact connected symplectic manifold with fixed points is necessarily Hamiltonian; see also [3, Corollary 3.9]. If $n = 2$ this result can be checked explicitly from the classification of symplectic 4-manifolds with symplectic 2-torus actions given in [20, Theorem 8.2.1] (since cases 2–5 in the statement of the theorem are shown not to be Hamiltonian; the only non-Kähler cases are given in items 3 and 4 as proved in [4, Theorem 1.1]).

If G is a Lie group with Lie algebra \mathfrak{g} acting symplectically on the symplectic manifold (M, ω) , the action is said to be *cohomologically free* if the Lie algebra homomorphism $\xi \in \mathfrak{g} \mapsto [\mathbf{i}_{\xi_M} \omega] \in H^1(M, \mathbb{R})$ is injective; $H^1(M, \mathbb{R})$ is regarded as an abelian Lie algebra. Ginzburg [8, Proposition 4.2] showed that if a torus $\mathbb{T}^k = (S^1)^k$, $k \in \mathbb{N}$, acts symplectically, then there exist subtori $\mathbb{T}^{k-r}, \mathbb{T}^r$ such that $\mathbb{T}^k = \mathbb{T}^r \times \mathbb{T}^{k-r}$, the \mathbb{T}^r -action is cohomologically free, and the \mathbb{T}^{k-r} -action is Hamiltonian. This homomorphism is the obstruction to the existence of a momentum map: it vanishes if and only if the action admits a momentum map. For compact Lie groups the previous result holds only up to coverings. If G is a compact Lie group, then it is well-known that there is a finite covering $\mathbb{T}^k \times K \rightarrow G$, where K is a semisimple compact Lie group. So there is a symplectic action of $\mathbb{T}^k \times K$ on (M, ω) . The K -action is Hamiltonian, since K is semisimple. The previous result applied to \mathbb{T}^k implies that there is a finite covering $\mathbb{T}^r \times (\mathbb{T}^{k-r} \times K) \rightarrow G$ such that the $(\mathbb{T}^{k-r} \times K)$ -action is Hamiltonian and the \mathbb{T}^r -action is cohomologically free; this is [8, Theorem 4.1]. The Lie algebra of $\mathbb{T}^{k-r} \times K$ is $\ker(\xi \mapsto [\mathbf{i}_{\xi_M} \omega])$.

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2 Proofs of Theorem 1 and Corollary 5

The proof extends Frankel's method [6] to the case of non-compact manifolds.

Let G be a compact Lie group acting on the symplectic manifold (M, ω) by means of symplectic diffeomorphisms. Let (ω, g, \mathbf{J}) be a G -invariant compatible triple. Let λ be a measure on M such that the Radon-Nikodym derivative of the Riemannian measure with respect to λ is a bounded function on M and denote by δ_λ the formal adjoint of \mathbf{d} relative to the L_λ^2 inner product, that is,

$$\int_M \langle \mathbf{d}\alpha, \beta \rangle d\lambda = \int_M \langle \alpha, \delta_\lambda \beta \rangle d\lambda,$$

for all $\alpha \in \Omega^q(M)$, $\beta \in \Omega^{q+1}(M)$, where $\langle \cdot, \cdot \rangle$ is the inner product on forms. Let $\|\cdot\|_{L_\lambda^2}$ be the L^2 -norm on forms relative to the measure λ .

By assumption we have:

- (i) Any $\alpha \in \Omega^1(M)$ such that $\|\alpha\|_{L_\lambda^2} < \infty$ and $\mathbf{d}\alpha = 0$ has a unique L_λ^2 -orthogonal decomposition $\alpha = \mathbf{d}f + \chi$, where $f \in C^\infty(M)$, $\mathbf{d}f \in L_\lambda^2(M)$, $\mathbf{d}\chi = 0$, $\delta_\lambda \chi = 0$, $\chi \in L^2(\wedge^1 M, g) \cap \Omega^1(M)$. Such forms χ are called *harmonic*. Let \mathcal{H} denote the space of harmonic one-forms.
- (ii) If a cohomology class $[\alpha] \in H^1(M, \mathbb{R})$ with $\|\alpha\|_{L_\lambda^2} < \infty$ has a harmonic representative, it is necessarily unique.
- (iii) $\mathbf{J}\mathcal{H} \subset \mathcal{H}$.

Remark 6 Condition (ii) can be reformulated as:

- (ii') If $f \in C^\infty(M)$, $\|\mathbf{d}f\|_{L_\lambda^2} < \infty$, and $\delta_\lambda \mathbf{d}f = 0$ then f is a constant function on each connected component of M .

Indeed, suppose (ii') holds and let α and β be two harmonic representatives of the same cohomology class with finite L_λ^2 -norm, then $\alpha - \beta = \mathbf{d}f$ for some $f \in C^\infty(M)$, $\|\mathbf{d}f\|_{L_\lambda^2} < \infty$. Therefore $\delta_\lambda \mathbf{d}f = \delta_\lambda(\alpha - \beta) = 0$ and hence, by (ii'), it follows that f is constant on each connected component of M implying that $\alpha = \beta$. Conversely, if $\|\mathbf{d}f\|_{L_\lambda^2} < \infty$ and $\delta_\lambda \mathbf{d}f = 0$, then $\mathbf{d}f$ is a smooth L_λ^2 harmonic one-form representing the zero cohomology class. Thus, by (ii), f is constant on each connected component of M . \circ

We want to show that if the G -action has fixed points on every connected component of M , then the action is Hamiltonian.

Proof of Theorem 1. We divide the proof in four steps.

Step 1 (Vanishing of harmonic one-forms along infinitesimal generators). We show that if $\alpha \in \Omega^1(M)$ is harmonic and $\|\alpha\|_{L_\lambda^2} < \infty$, then $\mathcal{L}_{\xi_M} \alpha = 0$ which is a standard result for the usual codifferential (Killing vector fields preserve harmonic one-forms). Since, in our case, we use δ_λ instead of the usual codifferential we give the proof.

We begin with the following observation: if $\varphi : M \rightarrow M$ is an isometry and preserves the measure $d\lambda$, that is, $\varphi^* g = g$ and $\varphi^*(d\lambda) = d\lambda$, then

$$\varphi^* (\langle \nu, \rho \rangle d\lambda) = \langle \varphi^* \nu, \varphi^* \rho \rangle d\lambda \quad (1)$$

for any $\nu, \rho \in \Omega^1(M)$.

Next, let $F_t := \Phi_{\exp(t\xi)}$ be the flow of ξ_M which is an isometry of (M, g) . Since $d\alpha = 0$ it follows that $dF_t^* \alpha = F_t^* d\alpha = 0$. We now show that F_t commutes with δ_λ . Indeed, for any $\beta, \gamma \in \Omega^1(M)$, we have

$$\begin{aligned} \langle \delta_\lambda F_t^* \beta, \gamma \rangle_{L_\lambda^2} &= \int_M \langle F_t^* \beta, d\gamma \rangle d\lambda \stackrel{(1)}{=} \int_M F_t^* (\langle \beta, (F_t)_* d\gamma \rangle) d\lambda = \int_M \langle \beta, (F_t)_* d\gamma \rangle d\lambda \\ &= \int_M \langle \delta_\lambda \beta, (F_t)_* \gamma \rangle d\lambda \stackrel{(1)}{=} \int_M (F_t)_* (\langle F_t^* \delta_\lambda \beta, \gamma \rangle) d\lambda = \langle F_t^* \delta_\lambda \beta, \gamma \rangle_{L_\lambda^2} \end{aligned}$$

and hence $\delta_\lambda F_t^* \beta = F_t^* \delta_\lambda \beta$. In particular, $\delta_\lambda \alpha = 0$ implies that $\delta_\lambda F_t^* \alpha = F_t^* \delta_\lambda \alpha = 0$, which shows that $F_t^* \alpha$ is harmonic.

However, in $H^1(M, \mathbb{R})$ we have $[F_t^* \alpha] = F_t^* [\alpha] = [\alpha]$ since F_t is isotopic to the identity; here F_t^* denotes the isomorphism induced by the diffeomorphism F_t on the cohomology groups. Therefore, the relation $[F_t^* \alpha] = [\alpha]$ implies that $F_t^* \alpha = \alpha$ since both $F_t^* \alpha$ and α are harmonic and each cohomology class has a unique harmonic representative by hypothesis (ii). Taking the t -derivative implies that $\mathcal{L}_{\xi_M} \alpha = 0$, as required.

Step 2 (Using the existence of fixed points). Define $\xi_M^\flat := g(\xi_M, \cdot) \in \Omega^1(M)$. If $\alpha \in \Omega^1(M)$ is harmonic and $\|\alpha\|_{L_\lambda^2} < \infty$, it follows from Step 1 that $0 = \mathcal{L}_{\xi_M} \alpha = \mathbf{i}_{\xi_M} d\alpha + d\mathbf{i}_{\xi_M} \alpha = d\mathbf{i}_{\xi_M} \alpha$. Thus $\alpha(\xi_M)$ is constant on each connected component of M . At this point we use the crucial hypothesis that the group action has at least one fixed point on each connected component. Thus, $\alpha(\xi_M) = 0$ on M . Therefore,

$$\langle \xi_M^\flat, \alpha \rangle_{L_\lambda^2} = \int_M \alpha(\xi_M) d\lambda = 0 \quad (2)$$

for any harmonic one-form α satisfying $\|\alpha\|_{L_\lambda^2} < \infty$, where $\dim M = 2n$.

Step 3 (Applying the existence of a Hodge decomposition). Since $d\mathbf{i}_{\xi_M} \omega = 0$ and $\|\mathbf{i}_{\xi_M} \omega\|_{L_\lambda^2} < \infty$, by hypothesis (i) we have $\mathbf{i}_{\xi_M} \omega = df^\xi + \chi^\xi$, where $f^\xi \in C^\infty(M)$, $\chi^\xi \in \Omega^1(M)$ is harmonic, $\|df^\xi\|_{L_\lambda^2} < \infty$, and $\|\chi^\xi\|_{L_\lambda^2} < \infty$.

Let us prove that $\chi^\xi = 0$. To do this, recall that \mathbf{J} is defined on one-forms by the relation $(\mathbf{J}\beta)(X) = \beta(\mathbf{J}X)$ for $\beta \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$. Therefore, $\mathbf{i}_{\xi_M} \omega = -\mathbf{J}\xi_M^\flat$. Indeed, for any $Y \in \mathfrak{X}(M)$ we have

$$(\mathbf{i}_{\xi_M} \omega)(Y) = \omega(\xi_M, Y) = -\omega(\xi_M, \mathbf{J}(\mathbf{J}Y)) = -g(\xi_M, \mathbf{J}Y) = -\xi_M^\flat(\mathbf{J}Y) = -(\mathbf{J}\xi_M^\flat)(Y).$$

Let $\alpha \in \Omega^1(M)$ be an arbitrary harmonic one-form such $\|\alpha\|_{L_\lambda^2} < \infty$. Then

$$\langle \mathbf{i}_{\xi_M} \omega, \alpha \rangle_{L_\lambda^2} = \langle -\mathbf{J}\xi_M^\flat, \alpha \rangle_{L_\lambda^2} = - \int_M \langle \mathbf{J}\xi_M^\flat, \alpha \rangle d\lambda = - \int_M \langle \xi_M^\flat, \mathbf{J}\alpha \rangle d\lambda = - \langle \xi_M^\flat, \mathbf{J}\alpha \rangle_{L_\lambda^2}.$$

By hypothesis (iii), $\mathbf{J}\alpha$ is harmonic and therefore $\langle \xi_M^b, \mathbf{J}\alpha \rangle_{L_\lambda^2} = 0$ by (2). Using again hypothesis (i), we conclude that $\chi^\xi = 0$. Therefore $\mathbf{i}_{\xi_M}\omega = \mathbf{d}f^\xi$ for any $\xi \in \mathfrak{g}$ and both sides of this identity are linear in $\xi \in \mathfrak{g}$.

Step 4 (Construction of an equivariant momentum map). Using a basis $\{e_1, \dots, e_r\}$ of \mathfrak{g} , we define a new map $\mu : M \rightarrow \mathfrak{g}^*$ by $\mu^\xi := \xi^1 f^{e_1} + \dots + \xi^r f^{e_r}$, where $\xi = \xi^1 e_1 + \dots + \xi^r e_r$. We clearly have $\mathbf{i}_{\xi_M}\omega = \mathbf{d}\mu^\xi$ which proves that $\mu : M \rightarrow \mathfrak{g}^*$, defined by the requirement that its ξ -component is μ^ξ for each $\xi \in \mathfrak{g}$, is a momentum map of the G -action.

Since G is compact, one can construct out of μ an equivariant momentum map (see, e.g., [17, Theorem 11.5.2], which shows that the action is Hamiltonian thereby completing the proof of the theorem.

Proof of Corollary 5. By hypothesis, M is a Kähler G -space, that is, (ω, g, \mathbf{J}) is a G -invariant compatible triple. Recall that $\omega^n = n!\mu_g$, where μ_g is the volume form associated to the Riemannian metric g (see, e.g., [2, formula (4.20)] and hence $L_{\mu_g}^2 = L_{\omega^n}^2$. Take $\lambda = \mu_g$ and hence $\delta_\lambda = \delta$ is the usual codifferential associated to the Riemannian metric g .

Now repeat the proof of Theorem 1. In Step 1 the crucial fact was that if a cohomology class has a harmonic L^2 representative, then it is unique. In the hypotheses of the corollary, this is implied by the weak L^2 -Hodge decomposition which holds for all complete non-compact Riemannian manifolds (see [14] for the general L^p formulation; for the strong L^p version see [13]) and the fact that every infinitesimal generator is a Killing vector field. Step 2 is unchanged. Step 3 follows again from the weak L^2 -Hodge decomposition. Indeed, by hypothesis, the smooth closed one-form $\mathbf{i}_{\xi_M}\omega \in L_{\mu_g}^2$ for any $\xi \in \mathfrak{g}$ and hence it decomposes $L_{\mu_g}^2$ -orthogonally as $\mathbf{i}_{\xi_M}\omega = \mathbf{d}f^\xi + \chi^\xi$, where $f^\xi \in C^\infty(M)$ and $\chi^\xi \in \Omega^1(M)$ is harmonic, $\|\mathbf{d}f^\xi\|_{L_{\mu_g}^2} < \infty$, $\|\chi^\xi\|_{L_{\mu_g}^2} < \infty$. As before, $\mathbf{i}_{\xi_M}\omega = \mathbf{J}\xi_M^b$ and for any harmonic $\alpha \in \Omega^1(M)$, $\|\alpha\|_{L_{\mu_g}^2} < \infty$, we have $\langle \mathbf{i}_{\xi_M}\omega, \alpha \rangle_{L_{\mu_g}^2} = -\langle \xi_M^b, \mathbf{J}\alpha \rangle_{L_{\mu_g}^2}$. Since M is Kähler, $\mathbf{J}\alpha$ is also harmonic (see, e.g., [23, Cor 4.11, Ch. 5]). Thus, by Step 2, $\langle \xi_M^b, \mathbf{J}\alpha \rangle_{L_\lambda^2} = 0$, which shows that $\chi^\xi = 0$. Step 4 is unchanged.

3 Examples

The previous results apply in the following examples.

3.1 Kähler quotients

A large class of examples can be obtained using a construction that will be presented below. In all that follows we assume that the manifolds are second countable. We begin with a few preliminary remarks.

Let Γ be a group that acts properly discontinuously on a manifold M , that is, each $x \in M$ has a neighborhood U such that $(\gamma \cdot U) \cap U = \emptyset$ for all $\gamma \neq e$. In particular, the Γ -action is free. Then the orbit space M/Γ is a smooth manifold and the projection $p : M \rightarrow M/\Gamma$ is both a local diffeomorphism and a covering map. Assume that $\mu \in \Omega^n(M)$, $n = \dim M$, is a Γ -invariant volume form on M . Then M/Γ has a unique volume form $\nu \in \Omega^n(M/\Gamma)$ such that $p^*\nu = \mu$. It is worth noting that the measure $m_{M/\Gamma}$ on M/Γ associated to the volume form ν does not coincide with the push forward of the measure m_M on M associated to the volume form μ . Recall that m_M is defined by the requirement that $\int_M \psi \, dm_M = \int_M \psi \mu$ for any $\psi \in C^\infty(M)$ with compact support, where the integral on the left is relative to the measure m_M and the integral on the right is relative to the volume form μ . A similar definition holds for $m_{M/\Gamma}$.

Let F be a fundamental domain of the Γ -action on M , that is, $F \subset M$ is a set such that each Γ -orbit intersects it in a single point. In particular, the restriction of the projection p to F gives a bijection between F and M/Γ . Assume:

- (i) the interior $\text{int}(F) \neq \emptyset$

(ii) the point set theoretical boundary $\overline{F} \setminus \text{int}(F)$ has zero m_M -measure.

Claim 1: For any $m_{M/\Gamma}$ -integrable $f \in C^\infty(M/\Gamma)$ we have

$$\int_{M/\Gamma} f \, dm_{M/\Gamma} = \int_{\text{int}(F)} (f \circ p) \, dm_M = \int_{\overline{F}} (f \circ p) \, dm_M. \quad (3)$$

Note that these integrals are *not* identically equal to zero by hypothesis (i).

To see this we begin by showing that the $m_{M/\Gamma}$ -measure of $M/\Gamma \setminus p(\text{int}(F))$ is zero. Indeed, since $p(\overline{F}) = M/\Gamma$, because F is a fundamental domain, and $p(\overline{F}) \setminus p(\text{int}(F)) \subseteq p(\overline{F} \setminus \text{int}(F))$ we get $m_{M/\Gamma}(M/\Gamma \setminus p(\text{int}(F))) = m_{M/\Gamma}(p(\overline{F}) \setminus p(\text{int}(F))) \leq m_{M/\Gamma}(p(\overline{F} \setminus \text{int}(F)))$. However, $m_M(\overline{F} \setminus \text{int}(F)) = 0$ by hypothesis (ii) and since the smooth map p maps measure zero sets to measure zero sets (M is second countable), it follows that $m_{M/\Gamma}(p(\overline{F} \setminus \text{int}(F))) = 0$.

Formula (3) follows from a change of variables and the remark above. Indeed, for any $m_{M/\Gamma}$ -integrable $f \in C^\infty(M/\Gamma)$ we have

$$\begin{aligned} \int_{M/\Gamma} f \, dm_{M/\Gamma} &= \int_{p(\text{int}(F))} f \, dm_M = \int_{p(\text{int}(F))} f \nu = \int_{\text{int}(F)} p^*(f \nu) \\ &= \int_{\text{int}(F)} (p^* f)(p^* \nu) = \int_{\text{int}(F)} (f \circ p) \mu = \int_{\text{int}(F)} (f \circ p) \, dm_M. \end{aligned}$$

This concludes the proof of Claim 1.

Assume now that a group Γ acts on two volume manifolds M_1 and M_2 and that the action on M_2 is properly discontinuous. The diagonal Γ -action on $M_1 \times M_2$ is also properly discontinuous. Let $F \subseteq M_2$ be the fundamental domain of the Γ -action on M_2 and assume (i) and (ii). The fundamental domain of the diagonal action is easily verified to equal $M_1 \times F$. Denote the measures on M_1 , M_2 , and $(M_1 \times M_2)/\Gamma$ by m_1 , m_2 , and q , respectively.

Claim 2: Assume that $m_2(\overline{F}) < \infty$. Then, for any q -integrable $f \in C^\infty((M_1 \times M_2)/\Gamma)$ such that $f \circ p$ does not depend on M_2 , we have

$$\int_{(M_1 \times M_2)/\Gamma} f \, dq = m_2(\overline{F}) \int_{M_1} (f \circ p) \, dm_1. \quad (4)$$

Indeed, denoting by $\chi_{\overline{F}}$ the characteristic function of \overline{F} , using the Fubini theorem we get

$$\begin{aligned} \int_{(M_1 \times M_2)/\Gamma} f \, dq &\stackrel{(3)}{=} \int_{M_1 \times \overline{F}} (f \circ p) \, d(m_1 \times m_2) = \int_{M_1 \times M_2} \chi_{\overline{F}}(f \circ p) \, d(m_1 \times m_2) \\ &= \int_{M_1} \left[\int_{M_2} \chi_{\overline{F}}(f \circ p) \, dm_2 \right] dm_1 = \int_{M_1} (f \circ p) \left[\int_{M_2} \chi_{\overline{F}} \, dm_2 \right] dm_1 \\ &= m_2(\overline{F}) \int_{M_1} (f \circ p) \, dm_1. \end{aligned}$$

Now we construct a class of examples for Corollary 5. Let a compact Lie group G act by Kähler transformations on a compact Kähler manifold M_1 . Suppose that the G -action on M_1 has fixed points. Let M_2 be a complete (possibly noncompact) Kähler manifold with finite volume. Let $\rho: \pi_1(M_2) \times M_1 \rightarrow M_1$ be a Kähler action which commutes with the G -action on M_1 . Let $\pi: \pi_1(M_2) \times M_2 \rightarrow M_2$ be the natural action by covering translations of $\pi_1(M_2)$ which is Kähler. Then the diagonal action $\pi \times \rho: \pi_1(M_2) : M_1 \times \widetilde{M}_2 \rightarrow M_1 \times M_2$ is also Kähler and commutes with the G -action. Let the twisted product $\mathcal{M} := (M_1 \times \widetilde{M}_2)/(\pi \times \rho)$ be equipped with the Kähler structure inherited from the product symplectic form on

$M_1 \times \widetilde{M}_2$. Denote by ω the symplectic form on \mathcal{M} . Let G act on \mathcal{M} by means of the G -action on M_1 , leaving M_2 fixed. Then the G -action on the complete Kähler manifold \mathcal{M} has fixed points. Since M_1 is compact and $m_2(\overline{F}) < \infty$, we have

$$\|\mathbf{i}_{\xi_{\mathcal{M}}}\omega\|_{L^2} = \int_{(M_1 \times \widetilde{M}_2)/(\pi \times \rho)} \|\mathbf{i}_{\xi_{\mathcal{M}}}\omega\|^2 dq \stackrel{(4)}{=} m_2(\overline{F}) \int_{M_1} (\|\mathbf{i}_{\xi_{\mathcal{M}}}\omega\|^2 \circ p) dm_1 < \infty. \quad (5)$$

Hence the assumption of Corollary 5 are satisfied and hence the G -action on \mathcal{M} is Hamiltonian.

Another way this example could have been treated is by applying Frankel's original theorem for compact Kähler manifolds to M_1 and then using reduction.

Let us spell out a concrete example of this situation. Let $M_1 := S^2$ with the standard Fubini-Study form and the standard Hamiltonian S^1 -action with 2^n fixed points. Let $M_2 := \Sigma_\infty$ be a complete Kähler symplectic 2-manifold of infinite genus and finite symplectic volume. Let $\pi_1(\Sigma_\infty)$ act on $\widetilde{\Sigma}_\infty$ by covering translations and let $\pi \times \rho$ act on $S^2 \times \widetilde{\Sigma}_\infty$ by the diagonal action, i.e. $\pi \times \rho: \pi_1(\Sigma_\infty) \rightarrow \mathrm{SO}(3) \times \mathrm{Symp}(\widetilde{\Sigma}_\infty)$, $g \mapsto (\pi(g), \rho(g))$, where $\rho: \pi_1(\Sigma) \rightarrow \mathrm{SO}(3)$ is any representation that commutes with the Hamiltonian S^1 -action on S^2 given by rotations about the vertical axis, in other words, that ρ factors through $\mathrm{SO}(2)$. This action is by Kähler automorphisms, so the quotient $(S^2 \times \widetilde{\Sigma}_\infty)/(\pi \times \rho)$ is Kähler.

3.2 Manifolds constructed by symplectic sum

One can also construct examples that satisfy the assumption of Corollary 5 using Gompf's symplectic sum [9]. These examples can be viewed as a sub-collection of those given in Section 3.1, so we do not provide details. Nonetheless, this construction is different, so it is worth presenting and outline. Let $k > 1$ and ω_{FS} be the Fubini-Study symplectic form on \mathbb{CP}^1 and let $\omega_{\mathbb{T}^2}$ be the standard area form on the 2-torus. Let the finite additive group $\mathbb{Z}/q\mathbb{Z}$, $q \in \mathbb{N}$, act on $(\mathbb{CP}^1)^{n-1}$ by rotating $360/q$ degrees on one or more copies of \mathbb{CP}^1 inside of $(\mathbb{CP}^1)^{n-1}$, and by rotations of $360/q$ degrees on the first (or both) standard sub-circles of \mathbb{T}^2 . This gives rise to a diagonal symplectic action of $\mathbb{Z}/q\mathbb{Z}$ on the product manifold $(\mathbb{CP}^1)^{n-1} \times \mathbb{T}^2$ equipped with the symplectic form $(n-1)\omega_{\mathrm{FS}} \oplus \frac{1}{m^k} \omega_{\mathbb{T}^2}$. The quotient $(\mathbb{CP}^1)^{n-1} \times_{\mathbb{Z}/q\mathbb{Z}} \mathbb{T}^2$ is a smooth manifold endowed with the quotient symplectic form denoted by $(n-1)\omega_{\mathrm{FS}} \oplus_q \frac{1}{m^k} \omega_{\mathbb{T}^2}$.

Consider any Hamiltonian S^1 -action on $(\mathbb{CP}^1)^{n-1}$, for example the action that acts in the usual way on the first component of the product, and acts trivially on the other components. Assume that S^1 acts trivially on the second factor \mathbb{T}^2 . This gives rise to a S^1 -action on $(\mathbb{CP}^1)^{n-1} \times \mathbb{T}^2$. Let x_m ($m \in \mathbb{N}$) be arbitrary distinct points in \mathbb{T}^2 . Let $\pi(Y_m)$ be the S^1 -invariant codimension-two symplectic submanifold of $(\mathbb{CP}^1)^{n-1} \times_{\mathbb{Z}/q\mathbb{Z}} \mathbb{T}^2$ given by projecting $(\mathbb{CP}^1)^{n-1} \times \{x_m\}$ under the canonical projection map $\pi: (\mathbb{CP}^1)^{n-1} \times \mathbb{T}^2 \rightarrow (\mathbb{CP}^1)^{n-1} \times_{\mathbb{Z}/q\mathbb{Z}} \mathbb{T}^2$. The S^1 -action on $(\mathbb{CP}^1)^{n-1} \times \mathbb{T}^2$ gives rise to a S^1 -action on $(\mathbb{CP}^1)^{n-1} \times_{\mathbb{Z}/q\mathbb{Z}} \mathbb{T}^2$, with “large” fixed point sets, and with even larger fixed point sets on the (infinite) connected symplectic sum

$$\mathcal{M}_k := \#_{\pi(Y_m), m \in \mathbb{N}} \left((\mathbb{CP}^1)^{n-1} \times_{\mathbb{Z}/q\mathbb{Z}} \mathbb{T}^2, (n-1)\omega_{\mathrm{FS}} \oplus_q \frac{1}{m^k} \omega_{\mathbb{T}^2} \right). \quad (6)$$

Let ω be the symplectic form of \mathcal{M}_k . The space \mathcal{M}_k is symplectomorphic to $(\mathbb{CP}^1)^{n-1} \times \Sigma_\infty$, where Σ_∞ is an infinite genus surface and has an area form that “decays” at infinity. Thus the manifold \mathcal{M}_k is non-compact and Kähler. By choosing an appropriate complete metric on Σ_∞ , \mathcal{M}_k is complete. So Corollary 5 applies.

3.3 Hodge decomposition with decay hypotheses

For non-compact manifolds the Hodge decomposition does not hold, in general. The first positive result is Kodaira's decomposition on a noncompact Riemannian manifold (see [12], [21, p. 165])

$$L^2(\wedge^q M, g) = \overline{\mathrm{d} C_0^\infty(\wedge^{q-1} M)} \oplus \overline{\delta C_0^\infty(\wedge^{q+1} M)} \oplus \mathcal{H}(\wedge^q M, g),$$

where δ is the L^2 formal adjoint of the exterior derivative operator \mathbf{d} , the closures are taken in L^2 , the subscript 0 denotes compact support, $\mathcal{H}(\wedge^q M, g) := \{\alpha \in L^2(\wedge^q M, g) \mid \mathbf{d}\alpha = 0, \delta\alpha = 0\}$, L^2 is taken relative to the measure associated to the Riemannian metric g , and the sum is L^2 -orthogonal. This kind of Hodge decomposition is not enough for our purposes since the best one can hope for, as the proof of Theorem 1 shows, is the existence of smooth functions f^ε with compact support. It is well known (see, e.g., [15]) that, in general, the Laplacian is not a Fredholm operator, that the kernel of the Laplacian does not consists of closed and coclosed forms, and that it does not give a unique or complete representation of the de Rham cohomology groups.

Improvements on this theorem are possible only if one changes the measure and the formal adjoint of \mathbf{d} or puts restrictions on the manifold and the metric. In both cases there are Hodge type decomposition theorems that fit the hypotheses of Theorem 1. We have already seen such a situation in the Corollary 5 when we used [13]. We will present further corollaries of Theorem 1 in both situations in this subsection and the next.

Ahmed and Stroock [1] conditions. Minimal requirements on the geometry of M that give a Hodge decomposition theorem and link harmonic forms to de Rham cohomology are given by Ahmed and Stroock in [1, §6]. The assumptions on the manifold (M, g) are:

(AM) *M is connected, complete, its Ricci curvature is bounded below by $-\kappa_{\text{Ric}} \leq 0$, and the Riemann curvature operator is bounded above, i.e., $\langle R\alpha, \alpha \rangle \leq \kappa \|\alpha\|_{L^2}^2$ for all $\alpha \in \Omega^2(M)$, where $\kappa \geq 0$.*

In addition, to get a Hodge decomposition, the Riemannian measure needs to be changed by a factor e^{-U} , where U satisfies certain conditions.

(AU) *Let $U : M \rightarrow [0, \infty)$ be a smooth function with the following properties:*

- *U has compact level sets*
- *There exists $C < \infty$ and $\theta \in (0, 1)$ such that $\Delta U \leq C(1 + U)$ and $\|\nabla U\|^2 \leq C e^{\theta U}$.*
- *There exists $\varepsilon > 0$ such that $\varepsilon U^{1+\varepsilon} \leq 1 + \|\nabla U\|^2$.*
- *There exists a $B < \infty$ such that $\langle v_x, (\nabla^2 U)(v_x) \rangle \geq -B\|v_x\|^2$ for all $x \in M$ and $v_x \in T_x M$.*

Let $\delta^U : \Omega^{k+1}(M) \rightarrow \Omega^k(M)$ be defined by $\delta^U \alpha := e^U \delta(e^{-U} \alpha)$, $\Delta^U = \delta^U \mathbf{d} + \mathbf{d} \delta^U$, $\mathcal{H}^U := \ker \Delta^U$, and $d\lambda_U := e^{-U} d\lambda(g)$, where $d\lambda(g)$ is the Riemannian measure associated to g . Under these conditions the Radon-Nikodym derivative e^{-U} of the Riemannian measure relative to $d\lambda_U$ is bounded on M . Relative to the measure $d\lambda_U$ one associates the L^2_U -spaces on forms.

We shall need the following results from [1]. Under the hypotheses **(AM)** and **(AU)** we have the following results:

- (1) *Any closed L^2_U -form $\alpha \in \Omega^1(M)$ has a unique L^2_U -orthogonal decomposition $\alpha = \mathbf{d}f + \chi$, where f is a L^2_U smooth function, $\chi \in \Omega^1(M)$, $\Delta^U \chi = 0$ and χ is of class L^2_U (see [1, Theorem 5.1].*
- (2) *Each cohomology class $[\alpha] \in H^1(M, \mathbb{R})$ has a unique smooth L^2_U representative in $\ker \Delta^U$ (see [1, Theorem 6.4]).*

Choose a G -invariant Riemannian metric and a G -equivariant complex structure J such that (ω, g, J) is compatible.

Corollary 7. *Assume that the compact Lie group acts on the connected symplectic manifold (M, ω) preserving the symplectic structure. Let (ω, g, J) be a compatible G -invariant triple and assume hypothesis **(AM)** holds. Let U be any G -invariant function satisfying **(AU)** and suppose that $J\mathcal{H}^U \subset \mathcal{H}^U$. If the G -action has fixed points, it is Hamiltonian.*

Proof. Let $\dim M = 2n$. We shall verify the conditions in Theorem 1. We show that δ^U is the formal adjoint of \mathbf{d} relative to the L^2_U inner product. Indeed, if $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$ are such that $\|\alpha\|_{L^2_U} < \infty$, $\|\beta\|_{L^2_U} < \infty$, $k \geq 0$, we have

$$\begin{aligned} \langle \alpha, \delta^U \beta \rangle_{L^2_U} &= \int_M \langle \langle \alpha, e^U \delta(e^{-U} \beta) \rangle \rangle e^{-U} d\lambda(g) = \int_M \langle \langle \alpha, \delta(e^{-U} \beta) \rangle \rangle d\lambda(g) \\ &= \int_M \alpha \wedge * \delta(e^{-U} \beta) = \int_M \mathbf{d}\alpha \wedge * e^{-U} \beta - \int_M \mathbf{d}(\alpha \wedge * e^{-U} \beta). \end{aligned}$$

The second terms vanishes by Stokes' Theorem since $\partial M = \emptyset$ and hence we get

$$\langle \alpha, \delta^U \beta \rangle_{L^2_U} = \int_M \langle \langle \mathbf{d}\alpha, e^{-U} \beta \rangle \rangle d\lambda(g) = \int_M \langle \langle \mathbf{d}\alpha, \beta \rangle \rangle e^{-U} d\lambda(g) = \langle \mathbf{d}\alpha, \beta \rangle_{L^2_U} \quad (7)$$

as required.

Next, we show that $\ker \Delta^U = \ker \mathbf{d} \cap \ker \delta^U$, which shows that a form is harmonic as defined in Theorem 1 (i) if and only if Δ^U vanishes on it. We begin with the identity

$$\int_M \mathbf{d}\alpha \wedge * e^{-U} \beta = \int_M \alpha \wedge * \delta(e^{-U} \beta)$$

which is equivalent to (7). If $\Delta^U \alpha = 0$, we have

$$\begin{aligned} 0 &= \int_M (\Delta^U \alpha \wedge * \alpha) e^{-U} d\lambda(g) = \int_M (\mathbf{d}\delta^U \alpha \wedge * \alpha) e^{-U} d\lambda(g) + \int_M (\delta^U \mathbf{d}\alpha \wedge * \alpha) e^{-U} d\lambda(g) \\ &= \int_M \langle \langle \mathbf{d}\delta^U \alpha, \alpha \rangle \rangle e^{-U} d\lambda(g) + \int_M \langle \langle \delta^U \mathbf{d}\alpha, \alpha \rangle \rangle e^{-U} d\lambda(g) = \langle \mathbf{d}\delta^U \alpha, \alpha \rangle_{L^2_U} + \langle \delta^U \mathbf{d}\alpha, \alpha \rangle_{L^2_U} \\ &\stackrel{(7)}{=} \langle \delta^U \alpha, \delta^U \alpha \rangle_{L^2_U} + \langle \mathbf{d}\alpha, \mathbf{d}\alpha \rangle_{L^2_U}. \end{aligned}$$

Therefore $\|\mathbf{d}\alpha\|_{L^2_U} = 0$ and $\|\delta^U \alpha\|_{L^2_U} = 0$ which implies that $\mathbf{d}\alpha = 0$ and $\delta^U \alpha = 0$ since α is smooth. This proves that $\ker \Delta^U \subset \ker \mathbf{d} \cap \ker \delta^U$. The converse inclusion is obvious.

This shows that the hypotheses of Theorem 1 are satisfied and hence, provided that the G -action has fixed points, we can conclude that it is Hamiltonian. \square

Gong and Wang [10] conditions. The Hodge decomposition holds also under different conditions involving the decay of the measure, as given in [10, Theorem 1.4]. Let the compact Lie group G act on the non-compact symplectic manifold (M, ω) by symplectic diffeomorphisms. Let (ω, g, \mathbf{J}) be a G -invariant compatible triple. Denote by \mathcal{R} the curvature term in the Weitzenböck formula on one-forms and assume that $e^V d\lambda(g)$ is a finite measure, where $d\lambda(g)$ is the Riemannian volume associated to g . Suppose that

- V is G -invariant
- $\mathcal{R} - \text{Hess}(V)$ is bounded below
- there exists a positive G -invariant function $U \in C^2(M)$ such that $U + V$ is bounded
- the sets $\{U \leq C\}$ are compact for all $C > 0$
- $\|\nabla U\| \rightarrow \infty$ as $U \rightarrow \infty$
- $\limsup_{U \rightarrow \infty} (\Delta U / \|\nabla U\|^2) < 1$

Then the Hodge decomposition holds, as shown in [10, Theorem 1.4]. Proceeding as before, we get the following result.

Corollary 8. *Assume that the compact Lie group acts on the connected symplectic manifold (M, ω) preserving the symplectic structure. Let (ω, g, \mathbf{J}) be a compatible G -invariant triple and assume the above hypotheses. If \mathbf{J} preserves the space of harmonic one-forms and the G -action has fixed points, it is Hamiltonian.*

3.4 Manifolds with two ends

Another class of examples is obtained by putting conditions on M . This class of examples appears in the work of Lockhart [15, Example 0.16]. We assume that M has finitely many ends, which means that M contains a compact submanifold M_0 whose smooth boundary ∂M_0 has finitely many components such that $M \setminus M_0 = \partial M_0 \times (0, +\infty)$. There is a natural additive monoid action of $[0, +\infty)$ on $\partial M_0 \times (0, +\infty)$ by translations on the right factor. We say that a metric on M is *translation invariant* if on $\partial M_0 \times (0, +\infty)$ is invariant under this action. Now suppose that h_∞ is a translation invariant metric, and let D_∞ denote the covariant derivative of the Levi-Civita connection associated to h_∞ . A metric h is *asymptotic to h_∞* if for each $k \in \mathbb{N}$ we have that

$$\lim_{z \rightarrow \infty} \sup_{\omega \in \partial M_0} \|D_\infty^k h(\omega, z) - D_\infty^k h_\infty(\omega)\|_{h_\infty} = 0.$$

The metric h is *asymptotically translation invariant* if h is asymptotic to a translation invariant metric. Now let h be an asymptotically invariant metric. Let $g = e^{-2\rho} h$ with $\rho \in C^\infty(M)$. We say that the metric g is *admissible* if there is a smooth, $(0, +\infty)$ -invariant 1-form θ on M_∞ with the property that

$$\lim_{z \rightarrow \infty} \sup_{\omega \in \partial M_0} \|D_{(h)}^{k+1} \rho - D_{(h)}^k \theta\|_h = 0,$$

where $D_{(h)}$ denotes the covariant derivative of the Levi-Civita connection associated to h .

Corollary 9. *Let (M, ω) be a connected symplectic manifold of dimension at least four which has two ends. Let (ω, g, \mathbf{J}) be a G -invariant compatible triple. Assume that $g = e^{2\rho} h$ is admissible, where h is an asymptotically translation invariant metric, $\rho(\omega, z)$ is decreasing, and $\rho(\omega, z) < -[(1 + \epsilon)/2] \ln z$ on $\partial M_0 \times [1, \infty)$ for some $\epsilon > 0$. If the G -action has fixed points, then it is Hamiltonian.*

The proof of the corollary follows from Theorem 1 because there is a Hodge decomposition theorem for these manifolds and each cohomology class in $H^1(M, \mathbb{R})$ has a unique harmonic representative, see [15, Formula (0.16.1)].

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